DIRECTED GRAPHS OVER TOPOLOGICAL SPACES: SOME SET THEORETICAL ASPECTS

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ABSTRACT
In this paper we shall consider problems of the following type. Suppose G is some set, $\mathfrak{U}$ is some family of subsests of G (e.g. G could be the Euclidean plane and $\mathfrak{U}$ might be the family of all sets of Lebesgue measure zero), and $\mathscr{G}$ is any directed graph over G (i.e. any collection of ordered pairs of members of G) such that for each $g \in G$ the set $\{h: \langle g, h\rangle \in \mathscr{G}\}$ belongs to the family $\mathfrak{U}$. How large a set $S \subseteq G$ must there exist with the property that $(S \times S) \cap \mathscr{G} = \emptyset$, i.e. such that it is totally disconnected? In many of the cases we shall consider (including the particular example above), the answer will turn out to be independent of the axioms of set theory and will remain so even after adjoining the negation of the continuum hypothesis.

1. Introduction

In this paper we shall consider problems of the following type. Suppose G is some set, $\mathfrak{U}$ is some family of subsets of G (e.g. G could be the Euclidean plane and $\mathfrak{U}$ might be the family of all sets of Lebesgue measure zero), and $\mathscr{G}$ is any directed graph over G (i.e. any collection of ordered pairs of members of G) such that for each $g \in G$ the set $\{h: \langle g, h\rangle \in \mathscr{G}\}$ belongs to the family $\mathfrak{U}$. How large a set $S \subseteq G$ must there exist with the property that $(S \times S) \cap \mathscr{G} = \emptyset$, i.e. such that it is totally disconnected? In many of the cases we shall consider (including the particular example above), the answer will turn out to be independent of the axioms of set theory and will remain so even after adjoining the negation of the continuum hypothesis.

In our proofs we shall not need to construct models of set theory and, in particular, we shall not make any explicit use of forcing techniques. Rather, we shall assume the axioms of Zermelo-Fraenkel set theory with choice (ZFC) and,

Received August 31, 1971

whenever necessary, cite known theorems concerning the existence of certain models of ZFC. Even here we shall normally be interested only in simple combinatorial properties of these models such as whether or not the continuum hypothesis holds or whether every set of reals of cardinality less than $2^{\aleph_0}$ has measure zero. We will then use these properties to prove theorems about graphs, and thus their consistency will imply the consistency of the theorems in question.

We shall need some notation. For any set A and any cardinal κ, we denote the cardinality of A by $|A|$; the cofinality of κ by $cf(\kappa)$; the smallest cardinal larger than κ by κ^+; the sets $\{B \subseteq A: |B| < \kappa\}$, $\{B \subseteq A: |B| \leqq \kappa\}$, and $\{B \subseteq A: |B| = \kappa\}$ by $[A]^{<\kappa}$, $[A]^{\leqq\kappa}$, and $[A]^{\kappa}$ respectively; the set $\{\langle a,b\rangle: \langle b,a\rangle \in A\}$ by A^{-1}; the set $\{\langle a,b\rangle \in A \times A: a \neq b\}$ by $\hat{A}$; and the cardinality of the power set of A by 2^A.

We shall identify cardinals with initial ordinals, i.e. a cardinal κ will be the set of all ordinals of cardinality less than κ. The cofinality of a cardinal κ may then be thought of as the smallest cardinal λ such that there exist cardinals $\kappa_0, \kappa_1, \cdots, \kappa_\alpha, \cdots$ $\alpha < \lambda$, each of cardinality less than κ but nevertheless satisfying $\sum_{\alpha<\lambda}\kappa_\alpha = \kappa$. It is well known that for any ordinal α we have $cf(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$ and that for any limit ordinal λ we have $cf(\aleph_\lambda) = cf(\lambda)$.

We shall use the symbols N, Z, and R to denote respectively the set of natural numbers, the set of all integers, and the set of all real numbers, and we shall denote a topological space over a set T with open sets $\mathcal{O}$ by $\mathcal{T}$ or $\langle T, \mathcal{O}\rangle$.

We define a *directed graph* over a set G to be any pair $\langle G, \mathcal{G}\rangle$ such that $\mathcal{G} \subseteq \hat{G}$, and for each point $g \in G$ we denote the set $\{h: \langle g,h\rangle \in \mathcal{G}\}$ by $\mathbf{g}_{\mathcal{G}}$ or, if $\mathcal{G}$ is clear from the context, by $\mathbf{g}$. A directed graph $\langle G, \mathcal{G}\rangle$ will be called *totally disconnected* or *free* iff $\mathcal{G} = \varnothing$ and *totally connected* iff $\mathcal{G} \cup \mathcal{G}^{-1} = \hat{G}$. For any set $H \subseteq G$ we define $\langle H, \mathcal{H}\rangle$ to be a *subgraph* of $\langle G, \mathcal{G}\rangle$ iff $\mathcal{H} = \hat{H} \cap \mathcal{G}$, and we shall frequently identify H and $\langle H, \hat{H} \cap \mathcal{G}\rangle$ when we are interested in both set properties of the one and graph properties of the other. Thus if we are dealing with a graph over R, we may speak of a set $H \subseteq R$ as being totally disconnected and of the second category. In this vein, if $\langle G, \mathcal{G}\rangle$ is a directed graph, then we define a set $H \subseteq G$ to be *$\mathcal{G}$-free* (or just *free* if $\mathcal{G}$ is clear from the context) iff $\hat{H} \cap \mathcal{G} = \varnothing$. We define two disjoint sets $A, B \subseteq G$ to be *$\mathcal{G}$-disjoint* iff $(A \times B) \cap (\mathcal{G} \cup \mathcal{G}^{-1}) = \varnothing$, and we define a family $\mathcal{F}$ of subsets of G to be *$\mathcal{G}$-free* (or again just *free*) iff its members are pairwise $\mathcal{G}$-disjoint. Finally, if $\mathfrak{U}$ is a family of subsets of G and $\langle G, \mathcal{G}\rangle$ is a directed graph such that for each $g \in G$ we have $\mathbf{g} \in \mathfrak{U}$, then we define $\langle G, \mathcal{G}\rangle$ to be an *$\mathfrak{U}$-graph*.

2. Combinatorial problems

The first problem of the kind we shall consider was posed by Paul Turán in a private communication in 1935 in which he asked, using somewhat different notation, if every $[R]^{<\aleph_0}$-graph admits an infinite free subgraph. This was answered affirmatively by Grünwald. Lázár showed that such a graph always admits a free subgraph of cardinality $2^{\aleph_0}$. Ruziewicz [12] then conjectured:

THEOREM 2.1. *Let S be an infinite set of cardinality κ, and let λ be any cardinal less than κ. Then every $[S]^{<\lambda}$-graph over S admits a totally-disconnected subgraph of cardinality κ.* ■

Special cases of this were proven by Sierpiński, Lázár, S. Picard, Fodor, and Erdös. Finally Hajnal [6] presented a proof of the theorem as stated.

In a sense this theorem is "best possible" as can be seen by the following which is found in [2].

THEOREM 2.2. *For every set S there exists a totally connected $[S]^{<|S|}$-graph $\langle S,\mathscr{G}\rangle$.*

PROOF. Choose any well-ordering $\prec$ of S such that $\langle S,\prec\rangle$ is order-isomorphic to $\langle |S|,\in\rangle$ and let $\mathscr{G} = \{\langle s,t\rangle : t \prec s\}$. ■

However, by introducing a notion which may be thought of as analogous to boundedness, we can obtain an intermediate result.

THEOREM 2.3. *Let S be any infinite set, let κ be any cardinal less than $|S|$, let $\mathscr{P} = \{S_\alpha : \alpha\in\kappa\}$ be any partition of S into κ disjoint subsets each of cardinality $|S|$, and let $\mathfrak{U} = \{T\in[S]^{<|S|} : \exists\,\alpha\in\kappa\,(T\subseteq \cup_{\beta<\alpha}S_\beta)\}$. Then:*

a. *There exists an $\mathfrak{U}$-graph $\langle S,\mathscr{G}\rangle$ which admits no infinite free subgraphs and no pairs U,V of $\mathscr{G}$-disjoint subsets both of cardinality $|S|$;*

b. *If $cf(\kappa) = cf(|S|)$, then there exists a totally connected $\mathfrak{U}$-graph $\langle S,\mathscr{H}\rangle$;*

c. *If $cf(\kappa) \neq cf(|S|)$, then every $\mathfrak{U}$-graph admits arbitrarily large finite free subgraphs.*

PROOF. We first note that since we are only interested in bounded subsets of $\mathscr{P}$, we may assume without loss of generality that κ is regular. For the remainder of this proof we set $\nu = |S|$, and we assume that we have indexed each S_α with ν so we have $S_\alpha = \{s^\alpha_\beta : \beta\in\nu\}$. We now continue with each section separately.

a. Define $\mathscr{G}$ to be the set $\{\langle s_\beta^\alpha, s_\delta \rangle \in \hat{S}: \gamma \leqq \alpha,\ \delta \leqq \beta\}$. Suppose now that $\{s_{\beta_i}^{\alpha_i}: i \in \omega\}$ is an infinite free set of distinct elements. From the definition of $\mathscr{G}$ we see immediately that $i \neq j \rightarrow \alpha_i \neq \alpha_j$, so we may assume without loss of generality that $i < j \rightarrow \alpha_i < \alpha_j$. But it also follows easily that $\alpha_i < \alpha_j \rightarrow \beta_i > \beta_j$, so $\{\beta_i: i \in \omega\}$ must be an infinite decreasing sequence of ordinals which is impossible.

Now let U and V be two non-empty $\mathscr{G}$-disjoint subsets of S. Let α be the least ordinal such that $(U \cup V) \cap S_\alpha \neq \varnothing$. We assume, by symmetry, that $U \cap S_\alpha \neq \varnothing$, and we let β be the least ordinal such that $s_\beta^\alpha \in U$. Then by $\mathscr{G}$-disjointness we have $V \subseteq \{s_\delta^\gamma: \delta < \beta\}$. But this latter set has cardinality $|\kappa \times \beta|$ which is strictly less than $|S|$.

b. Let $\{\nu_\alpha: \alpha < \kappa\}$ be a strictly increasing cofinal subset of ν, and for each $\alpha < \kappa$ let $T_\alpha = \{s_\gamma^\beta: \beta < \alpha \wedge \gamma < \nu_\alpha\}$. Also, let $\prec$ be any well ordering of S satisfying $t \in (T_\beta - T_\alpha) \wedge s \in T_\alpha \rightarrow s \prec t$. We note that since each T_α has cardinality less than ν, we have S order isomorphic under $\prec$ to ν, and thus the set of predecessors under $\prec$ of any given element has cardinality less than ν. Since it also follows that $T_\alpha \subseteq \cup \{S_\beta: \beta < \alpha\}$, we see that the graph $\langle S, \{\langle s, t \rangle: t \prec s\}\rangle$ is a totally-connected $\mathfrak{U}$-graph on S.

c. Our proof will essentially parallel the proof of Theorem 2 of Erdös and Hajnal [4, p. 188] except that we will have to allow for the possible singularity of ν. Thus let $\lambda = cf(\nu)$ and let $\langle S, \mathscr{G}\rangle$ be any $\mathfrak{U}$-graph over S. We shall prove by induction that for each natural number n and each ordinal $\alpha < \kappa$ there exists an n-element totally-disconnected subset T_n^α of S such that $T_n^\alpha \cap S_\alpha \neq \varnothing$, but $T_n^\alpha \cap \cup_{\beta<\alpha} S_\beta = \varnothing$. This is trivially true for $n = 1$, so we suppose it is true for $n = m$ and prove it for $n = m + 1$. Choose any ordinal α. By our induction hypothesis we may choose a family $\mathscr{S} = \{T_m^\beta: \alpha < \beta < \kappa\}$ such that each T_m^β is as described above. For each $T \in \mathscr{S}$ let $\mathbf{T} = \cup\{\mathbf{t}: t \in T\}$. Because each $T \in \mathscr{S}$ is finite, we have $|\mathbf{T}| < \nu$.

Now suppose $|\cup\{\mathbf{T}: T \in \mathscr{S}\}| = \nu$. Since $\mathscr{S}$ has cardinality κ, this implies that $\lambda = cf(\nu) \leqq \kappa$ and since $cf(\nu)$ cannot equal κ by the hypothesis of part c (this is the only place we use this hypothesis), we have $\lambda < \kappa$. Let $\nu^* = \{\nu_\beta: \beta < \lambda\}$ be any strictly increasing cofinal set of cardinals in ν, and for each $\beta \in \lambda$ let $C_\beta = \{\gamma \in \kappa: |\mathbf{T}_m^\gamma| < \nu_\beta\}$. Because $\kappa = \cup_{\beta<\lambda} C_\beta$, $\lambda < \kappa$, and κ is regular, there must be some $\delta < \kappa$ such that $|C_\delta| = \kappa$. Call this set κ^*.

On the other hand, if $|\cup\{\mathbf{T}: T \in \mathscr{S}\}|$ is strictly less than ν, set $\kappa^* = \kappa$. In either case we set $\mathscr{S}^* = \{T_m^\beta : \alpha < \beta \in \kappa^*\}$ and we note that κ^*, because it has cardinality κ, is still cofinal with κ. However, we may now assume that $\cup\{T: T \in \mathscr{S}^*\}$ has cardinality strictly less than ν. Thus we may choose a point $s \in (S_\alpha - \cup\{T: T \in \mathscr{S}^*\})$. But s is bounded; i.e. there exists a $\beta \in \kappa$ such that $s \subseteq \cup_{\gamma<\beta} S^\gamma$. Thus we may choose any ordinal $\delta \in \kappa^* - \beta$ and set $T_{m+1}^\alpha = \{s\} \cup T_m^\delta$. ■

In particular if $\mathfrak{U}^{BC}$ is the family of bounded members of $[R]^{<|R|}$, we have:

COROLLARY. 2.4. *Every $\mathfrak{U}^{BC}$-graph over R admits arbitrarily large finite free subsets, and there exists such a graph which does not admit any infinite free subsets.*

PROOF. In 2.3 let $\kappa = \omega$ and for each $n \in \omega$ let $S_n = (-n-1, -n] \cup [n, n+1)$. Since $|R|$ can never have cofinality ω, the result follows. ■

Finally, we shall need a generalization of the Ruziewicz conjecture.

THEOREM 2.5. *Let S be any infinite set, let $\mathscr{P}$ be any subfamily of $[S]^{|S|}$ of cardinality less than $|S|$, and let κ be any cardinal less than $|S|$. Then every $[S]^{<\kappa}$-graph admits a free set $F \subseteq S$ satisfying $P \in \mathscr{P} \to |F \cap P| = |S|$.* ■

This was first stated by Erdös and Fodor [3] under the additional assumption that $\mathscr{P}$ be a disjoint family, and proven for $|S|$ regular. Hajnal [6] notes that his proof of 2.1 also applies to the case $|S|$ singular of the Erdös-Fodor theorem, and it is easily seen that both proofs can be modified to cover the case $\mathscr{P}$ not disjoint.

3. Topological problems

We begin with some observations which follow immediately from our previous section and various known results concerning some of the models of ZFC which have been recently constructed. Although many of the results in this section will be stated with respect to R, they nevertheless can be generalized to Euclidean n-space for all $n \geqq 1$. For the remainder of this paper let $\mathfrak{U}^N$ be the family of nowhere-dense subsets of R, let $\mathfrak{U}^F$ be the family of subsets of R which are of the first category, and let $\mathfrak{U}^M$ be the family of subsets of R which have Lebesgue measure zero.

As $\mathfrak{U}^M$ and $\mathfrak{U}^F$ are easier to handle, we treat them first. Let $\mathfrak{U}^{MFC} = \mathfrak{U}^M \cap \mathfrak{U}^F \cap [R]^{<R}$. We first note:

THEOREM 3.1. *If the continuum hypothesis holds, then there exist totally-connected $\mathfrak{U}^{MFC}$-graphs over R.*

PROOF. The result follows immediately from 2.2 and the fact that every countable subset of R is of the first category and has measure zero. ■

However, the negation of the continuum hypothesis does not yield any information at all. The reason for this is that the proof of 3.1 does not really use the full continuum hypothesis but rather the following weaker hypothesis which we shall refer to as **C**.

Every subset of R of cardinality less than $2^{\aleph_0}$ is of the first category and has measure zero.

It is known that **C** is a consequence of an axiom due to Martin [10] which is itself known [14] to be consistent with the negation of the continuum hypothesis. Thus we have:

THEOREM 3.2. *It is consistent with the negation of the continuum hypothesis that there exist totally-connected $\mathfrak{U}^{MFC}$-graphs over R.* ■

In other models of ZFC the situation is strikingly different, and we may even pick up some information about $\mathfrak{U}^N$.

THEOREM 3.3. *It is consistent with the axioms of set theory (ZFC) that every $\mathfrak{U}^F$-graph (and therefore every $\mathfrak{U}^N$-graph) over R admits a free subgraph which is of the second category and has cardinality $2^{\aleph_0}$, and that there nevertheless exists a totally connected $\mathfrak{U}^M$-graph over R.*

PROOF. Suppose that to a countable standard model of ZFC we add a set G of at least $\aleph_2$ generic (i.e., we use conditions of the type first introduced by Cohen [1, chap. iv]) real numbers. Then it is well known [15] that in the new model, G has the property that for every set $A \in \mathfrak{U}^F$ the set $A \cap G$ is countable. (This follows from the fact that every member of $\mathfrak{U}^N$ is contained in a closed member of $\mathfrak{U}^N$ and that, in models of this type, every closed set of reals is constructible from a countable subset C of G. It can then be shown that no member of $G - C$ can possibly be forced to be a member of a nowhere dense set constructible from C.) Now suppose that $\langle R, \mathscr{G} \rangle$ is any $\mathfrak{U}^F$-graph. Because of the special property of G, the graph $\langle G, \hat{G} \cap \mathscr{G} \rangle$ is a $[G]^{<\aleph_1}$-graph and since G has cardinality $2^{\aleph_0} > \aleph_1 > \aleph_0$, there must, by 2.1, exist a free subset H of cardinality $2^{\aleph_0}$.

It is clear that H is also a $\mathscr{G}$-free subset, of R and because it is an uncountable subset of G, it must be of the second category.

Furthermore, Solovay has proven (private communication) that, in the above construction, each Borel set of measure zero which is in the model one begins with, can be extended to a Borel set of measure zero in the new model in such a way as to insure that every member of R is contained in at least one of these extensions. Now assume that the original model satisfied the continuum hypothesis. Then in the new model, R is contained in a union of $\aleph_1$ sets each of measure zero and therefore can be partitioned into a family $\mathscr{F} = \{M_\alpha : \alpha \in \aleph_1\}$ of disjoint members of $\mathfrak{U}^M$. Thus the graph $\langle R, \{\langle r,s\rangle \in \hat{R} : r \in M_\alpha \wedge s \in M_\beta \rightarrow \beta \leqq \alpha\}\rangle$ is a totally connected $\mathfrak{U}^M$-graph. ■

On the other hand we also have:

THEOREM 3.4. *It is consistent with the axioms of set theory that every $\mathfrak{U}^M$-graph over R admits a free set of cardinality $2^{\aleph_0}$ which is not of measure zero, and that there nevertheless exists a totally-connected $\mathfrak{U}^F$-graph.*

PROOF. Solovay has proven (private communication) that if one adds random reals [13] rather than generic reals to a countable standard model of ZFC, one obtains results analogous to those mentioned in the proof of 3.3 except that "measure zero" and "first category" must be interchanged. Thus if a set H of more than $\aleph_1$ random reals is added to a countable standard model of $ZFC + 2^{\aleph_0} = \aleph_1$, then H in the resulting model will be a subset of R such that for every set $A \in \mathfrak{U}^M$, the set $A \cap H$ will be countable. Furthermore, it will be possible to partition R into a union of $\aleph_1$ disjoint sets each of the first category. The proof then proceeds as in 3.3. ■

Now denote the bounded members of $\mathfrak{U}^N$, $\mathfrak{U}^F$, $\mathfrak{U}^M$, and $\mathfrak{U}^{FMC}$ by $\mathfrak{U}^{BN}$, $\mathfrak{U}^{BF}$, $\mathfrak{U}^{BM}$, and $\mathfrak{U}^{BFMC}$ respectively. Erdös and Hajnal [4] have proven that if $\mathfrak{U}$ is the family of *bounded* subsets of R of outer measure no greater than one, then every $\mathfrak{U}$-graph admits arbitrarily large finite free subgraphs. These authors have asked about infinite free subgraphs [4, and 5, #38C]. We note that it is consistent with the axioms of set theory that such infinite free subgraphs do not exist.

THEOREM 3.5. *Every $\mathfrak{U}^{BF}$-or $\mathfrak{U}^{BM}$-graph over R admits arbitrarily large finite free subgraphs. However,* **C** *(and therefore the continuum hypothesis) implies that there exists a $\mathfrak{U}^{BFMC}$-graph $\langle R, \mathscr{G}\rangle$ which admits no infinite free* *subgraphs and such that there do not exist two $\mathscr{G}$-disjoint subsets of R of cardinality* $2^{\aleph_0}$.

PROOF. The existence of arbitrarily large finite free subgraphs follows immediately from the proof of Erdös and Hajnal's theorem (or the proof of 2.3), while the existence of the graph $\langle R, \mathscr{G} \rangle$ follows directly from 2.3 by setting $\kappa = \aleph_0$ and letting $\mathscr{P} = \{(-n-1, -n] \cup [n, n+1) : n \in \omega\}$. ■

We do not know if the negation of the continuum hypothesis yields any positive information about $\mathfrak{U}^N$-graphs. Erdös [2] has proven that every $\mathfrak{U}^N$-graph admits an infinite free subset. Máté [11] has proven that every such graph admits free subsets of every countable well-order type, (under $<$). We shall prove (4.11) that every such graph admits free subsets of every countable order type, but, under the assumption of the negation of the continuum hypothesis, the closest we can come to proving these results "best possible" is:

THEOREM 3.6. *It is consistent with the negation of the continuum hypothesis that there exists an $\mathfrak{U}^{NC}$-graph over R which admits no free subgraphs of cardinality greater than $\aleph_1$.*

PROOF. In the models of ZFC constructed by Solovay which we have referred to, and in certain models constructed by the present author [8], the continuum can be made arbitrarily large, but R can always be partitioned into a family $\mathscr{F} = \{N_\alpha : \alpha \in \aleph_1\}$ of disjoint, nowhere-dense subsets. Let $\prec$ be any well ordering (in such a model) such that $\langle R, \prec \rangle$ has the order type of the cardinal $2^{\aleph_0}$ (again in the model). Then the graph $\langle R, \mathscr{G} \rangle$, where $\mathscr{G}$ is the set

$$\{\langle s, t \rangle \in \bigcup_{\alpha < \aleph_1} \hat{N}_\alpha : t \prec s\},$$

is an $\mathfrak{U}^{NC}$-graph which admits no free subgraphs of cardinality greater than $\aleph_1$. ■

Given the continuum hypothesis, however, we can show these theorems are "best possible" and much more. We can construct a $\mathfrak{U}^{BNC}$-graph over R admitting no uncountable free subgraphs. Moreover, for each $r \in R$ the set $\mathbf{r}$ is either empty or has order type ω under $<$, admits only r as a limit point, and is contained in the open interval $(r - \varepsilon_r, r)$ where ε_r may be any specified positive real number and may vary for different $r \in R$. This answers several questions raised by Erdös and Hajnal [2, p. 53; 4, pp. 188–189; 5, # 38A, p. 35]. That such a graph can be constructed follows immediately from:

THEOREM 3.7. *Assume the continuum hypothesis and let* $\mathscr{T} = \langle T, \mathscr{O} \rangle$ *be any second-countable Hausdorff space of cardinality* $2^{\aleph_0}$. *Let* $\mathfrak{U}$ *be the set of nowhere-dense subsets of* T, *let* $\prec$ *be any total ordering on* T *which admits no decreasing sequence* $\{t_\alpha \in T : \alpha \in \aleph_1\}$ *such that* $\alpha < \beta \to t_\alpha \succ t_\beta$, *and for each* $t \in T$ *let* $\{B^t_n : n \in \omega\}$ *be a nested base for the open neighborhoods of* t. *Then there exists an* $\mathfrak{U}$*-graph* $\langle T, \mathscr{G} \rangle$ *which admits no uncountable free subgraphs and for which* $\mathbf{t} \neq \varnothing$ *implies*:

a. $s \in \mathbf{t} \to s \prec t$,

b. $\mathbf{t} \subseteq B^t_0$,

c. $(B^t_n - B^t_{n+1}) \cap \mathbf{t}$ *has at most one element*,

d. $\mathbf{t}$ *has cardinality* $\aleph_0$,

e. t *is the unique limit point of every infinite subset of* $\mathbf{t}$,

f. $s \neq t \to \mathbf{s} \cap \mathbf{t}$ *is finite.*

PROOF. For convenience we identify T with $\aleph_1$. Let $\mathscr{B} = \{B_i : i \in \omega\}$ be a countable base for $\mathscr{O}$, and for any set $S \subseteq T$, let $S^* = \{s : s \in O \in \mathscr{O} \to \exists t \prec s$ $t \in O \cap S\}$. We first note that for any $S \subseteq T$, the set $S - S^*$ is countable. This follows from the fact that to each $s \in S - S^*$ we may assign a set $B_s \in \mathscr{B}$ such that $s \in B_s$ and $B_s \cap \{t \in S : t \prec s\} = \varnothing$. Since $\prec$ is a total ordering, we have $B_s = B_t \to s = t$; thus the countability of $\mathscr{B}$ implies the countability of $S - S^*$.

Now let $\{C_\alpha : \alpha \in \aleph_1\}$ be an enumeration of $[T]^{<\aleph_1}$, and for each $\alpha \in \aleph_1$ let $\mathscr{C}_\alpha = \{C_\beta : \beta < \alpha \in C^*_\beta\}$. If $\mathscr{C}_\alpha = \varnothing$, we set $\boldsymbol{\alpha} = \varnothing$. Otherwise we let $\{C^\alpha_n : n \in \omega\}$ be an enumeration, with repeats if necessary, of $\mathscr{C}_\alpha$, and we define $\boldsymbol{\alpha} = \{\alpha_n : n \in \omega\}$ inductively. Assume we have defined α_m for each $m < n$. Choose any k such that $\{\alpha_0, \cdots, \alpha_{n-1}\} \cap B^\alpha_k = \varnothing$, and let α_n be any member of $B^\alpha_k \cap C^\alpha_n \cap \{t : t \prec \alpha\}$.

We see immediately that $\boldsymbol{\alpha}$ satisfies a, b, c, and d, and, because $\mathscr{T}$ is Hausdorff, $\alpha_n \in \bigcap_{m<n} B^\alpha_m$, and $\{B^\alpha_m\}$ is a nested base; e and therefore f also follow.

To prove that T admits no uncountable free subgraphs is a bit more difficult. Let S be any uncountable subset of T. We shall prove that S is not free.

We first prove that for some $\alpha \in \aleph_1$ we have $C_\alpha \subseteq S$ and $S^* = C^*_\alpha$. For each $B_n \in \mathscr{B}$, let D_n be any countable subset of $B_n \cap S$ such that $a \in (B_n \cap S) \to \exists d \in D_n$ $(d \preceq a)$. Such a countable set D_n always exists because $\langle T, \prec \rangle$ admits no uncountable decreasing chains. Thus $\bigcup_{n \in \omega} D_n \in [T]^{<\aleph_1}$ and is therefore equal to C_α for some α. Now choose any $s \in S^*$ and any set O such that $s \in O \in \mathscr{O}$. Because $\mathscr{B}$ is a base, we have $s \in B_n \subseteq O$ for some $n \in \omega$, and therefore we have an element $t \in S \cap B_n$ such that $t \prec s$. This in turn implies the existence of an element

$d \in D_n \subseteq B_n \subseteq O$ such that $d \prec t \preceq s$. Thus since O was arbitrary, we have $s \in C_\alpha^*$, and since s was arbitrary, we have $S^* \subseteq C_\alpha^*$.

Finally, because S is uncountable while $S - S^*$ is countable, we may choose a point β such that $\alpha < \beta \in S^* \cap S$. Since $S^* \subseteq C_\alpha^*$, we have $\beta \in C_\alpha^*$ and therefore $C_\alpha \in \mathscr{C}_\beta$. But from our construction this yields $\emptyset \neq \boldsymbol{\beta} \cap C_\alpha \subseteq \boldsymbol{\beta} \cap S$, so S is indeed not free. However, S was an arbitrary uncountable subset of T, so we are done. ■

This theorem may be generalized to higher cardinals. Suppose we look at a Hausdorff space $\mathscr{T} = \langle T, \mathscr{O} \rangle$ of cardinality $\kappa^+ = 2^\kappa$ which admits a base $\mathscr{B}$ of cardinality κ. The order $\prec$ will now have to admit no decreasing sequence of order type κ^+, and one other property of $\mathscr{T}$ will be required. When constructing α_n, we assumed that there existed a k such that $B_k^\alpha \cap \{\alpha_\zeta : \zeta < n\} = \emptyset$. For n finite this followed from the fact that $\mathscr{T}$ was Hausdorff, but by 2.1 we see that we cannot hope to require each $\boldsymbol{t}$ to be countable. Instead we require that each $\boldsymbol{t}$ be either empty or of cardinality κ. Thus we must assume (and therefore require) that each $t \in T$ (with at most κ exceptions) admit, as an open base for its neighborhoods, a nested family $\mathscr{B}^t = \{B_\alpha^t : \alpha \in \kappa\}$ satisfying $S \in [T - \{t\}]^{<\kappa} \rightarrow \exists B \in \mathscr{B}^t (B \cap S = \emptyset)$.

Given these conditions we can construct a $\mathfrak{U}$-graph which admits no free subgraphs of cardinality κ^+ and for which $\boldsymbol{t} \neq \emptyset$ implies a, b, and c of 3.7 and

d^κ. $\boldsymbol{t}$ has cardinality κ,

e^κ. t is the unique limit point of every subset of $\boldsymbol{t}$ of cardinality κ,

f^κ. $s \neq t \rightarrow \boldsymbol{s} \cap \boldsymbol{t}$ has cardinality less than κ.

This has an interesting application. In [3] Erdös and Fodor ask the following: Let κ be any cardinal and let G be a $[\kappa]^{<\kappa}$-graph over κ such that for some cardinality $\lambda < \kappa$ each pair α, β of distinct ordinals in κ satisfies $|\boldsymbol{\alpha} \cap \boldsymbol{\beta}| < \lambda$. Does there then exist a free subgraph of cardinality κ? (They prove that there always exists an infinite free subgraph.) Hajnal has proven [7] that the answer is negative whenever κ is regular and $2^\kappa = \kappa^+$. We note that the generalization of 3.7 yields an alternate proof of a special case of this.

3.8. THEOREM. *Let κ be any regular cardinal such that $2^\kappa = \kappa^+$ and $2^\kappa = \kappa$. Then there exists a $[\kappa^+]^{\geqq\kappa}$-graph on κ^+ such that $\alpha < \beta < \kappa \rightarrow |\boldsymbol{\alpha} \cap \boldsymbol{\beta}| < \kappa$, but which contains no free subgraphs of cardinality κ^+.*

PROOF. We need only exhibit a topological space of cardinality κ^+ having the required properties and apply our generalization. Such a space can be obtained by generalizing the notion of the real line. Let F_κ be the set of functions from κ into $\{0,1\}$, let R_κ be the set $\{f \in F_\kappa : \alpha \in \kappa \to \exists\beta[\alpha < \beta < \kappa \wedge f(\beta) \neq f(\alpha)]\}$, let $\prec_\kappa$ be the lexicographical ordering of R_κ, and let $\mathcal{O}_\kappa$ be the order topology on R_κ induced by $\prec_\kappa$. It is easily seen that the set of open intervals whose endpoints are in $F_\kappa - R_\kappa$ forms a base of cardinality κ (because $2^{\kappa} = \kappa$) for $\mathcal{O}_\kappa$. Also, because κ is regular, the intersection of any nested sequence of fewer than κ open intervals is itself an open interval. ■

We may generalize 3.7 in another direction by allowing the continuum hypothesis to fail, but requiring instead that Martin's axiom hold. The result we obtain while not exact, is still interesting in the light of 4.8–4.10.

THEOREM 3.9. *Assume Martin's axiom. Let* $\mathcal{T} = \langle T, \mathcal{O}\rangle, \mathfrak{U}$, *and* $\prec$ *satisfy the hypotheses of* 3.7, *and for each* $t \in T$ *let* B^t *be any open set containing* t. *Then there exists an* $\mathfrak{U}$-*graph which admits no free subgraphs of cardinality* $2^{\aleph_0}$ *and which satisfies*:

a. $s \in \mathbf{t} \to s \prec t$, *and*

b. $\mathbf{t} \subseteq B^t$.

PROOF. We proceed exactly as in the proof of 3.7 with the same definitions, except for that of $\boldsymbol{\alpha}$. By 2.1 we cannot let some countable set equal $\boldsymbol{\alpha}$ and still expect to obtain a graph with no free subgraphs of cardinality $2^{\aleph_0}$. So we shall use a different construction based on Martin's axiom. The reader not familiar with this axiom should consult [10].

Choose any fixed $\alpha \in T$. If $\alpha \in (T - T^*)$ let $\boldsymbol{\alpha} = \emptyset$; otherwise let $\mathcal{B}_\alpha = \{B \in \mathcal{B} : \exists D \in \mathcal{O}[\alpha \in D \subseteq (T - B)]\}$, let $T_\alpha = \{t \prec \alpha : t \in T \cap B^\alpha\}$, and let $P = [\mathcal{B}_\alpha]^{<\aleph_0} \times [T_\alpha]^{<\aleph_0}$. For each $p = \langle a, b\rangle \in P$, let $p_0 = a$ and $p_1 = b$. We wish to think of $\cup p_0$ as that part of T which is removed so that the remainder will be nowhere dense, and p_1 as a subset (to be) of $\boldsymbol{\alpha}$. Thus we consider only the set $P^c = \{p \in P : p_1 \cap \cup p_0 = \emptyset\}$. Let $\mathcal{P}$ be the partial-order structure $\langle P^c, \prec\rangle$ defined by $p \prec q \to p_0 \Leftrightarrow q_0 \wedge p_1 \subseteq q_1$. We note that for any two elements $p, q \in P^c$, if $p_0 = q_0$, then the pair $r = \langle p_0, p_1 \cup q_1\rangle$ belongs to P^c and extends both of them, so $\mathcal{P}$ satisfies the countable chain condition. For each $B \in \mathcal{B}$ let

$$F_B = \{p \in P^c : \exists B' \in p_0[B' \subseteq B]\},$$

and for each β such that $C_\beta \in \mathscr{C}_\alpha$ let $F_\beta = \{p \in P^c: p_1 \cap C_\beta \neq \emptyset\}$. It is easily seen that each F_B and each F_β is dense open. So by Martin's axiom, there exists a filter G over $\mathscr{P}$ which is $(\{F_B: B \in \mathscr{B}\} \cup \{F_\beta: C_\beta \in \mathscr{C}_\alpha\})$-generic (i.e. $G \subseteq \mathscr{P}^c$, $p, q \in G \rightarrow \exists r \in G(p \prec r \wedge q \prec r)$, $B \in \mathscr{B} \rightarrow F_B \cap G \neq \emptyset$, and $C_\beta \in \mathscr{C}_\alpha \rightarrow F_\beta \cap G \neq \emptyset$). Thus if we set $\alpha = \cup \{p_1 : p \in G\}$, we see immediately that α satisfies a and b of the theorem, is nowhere dense, and intersects each $C_\beta \in \mathscr{C}_\alpha$. The remainder of the theorem now follows exactly as before. ■

4. Free families

We conclude by considering the existence of $\mathscr{G}$-free families of subsets of R where $\langle R, \mathscr{G} \rangle$ is a $\mathfrak{U}^N$-graph. Since we can always find an infinite free set and partition it into infinitely many disjoint infinite pieces which will, of course, be $\mathscr{G}$-disjoint, we will only be interested in families which are either themselves uncountable or have uncountable members. The former will not, in general, be possible because if we had such a family, then by choosing one representative from each member we could obtain an uncountable free set thus violating 3.7., at least if the continuum hypothesis holds. Thus in the general case, we are reduced to looking for free families some of whose members are uncountable. Ideally, we would like to find such families each of whose members has cardinality $2^{\aleph_0}$, but our proofs instead lead to families each of whose members is of the second category. Given **C** or therefore, the continuum hypothesis or Martin's axiom, we can immediately infer that these members actually do have cardinality $2^{\aleph_0}$; otherwise we do not know.

There is a related question which we shall consider first. Suppose we are given a family $\mathscr{E} = \{E_i: i \in I\}$ of "reasonable" disjoint sets. Can we find a free family $\mathscr{F} = \{F_i \subseteq E_i: i \in I\}$ of "large" sets? For $\mathscr{E}$ finite, the answer will be yes; for $\mathscr{E}$ infinite, we do not have complete information, even for "small" sets.

An obvious family to consider is the family $\mathscr{Z} = \{[z, z+1): z \in Z\}$. We do not know whether there always exists a free family $\mathscr{F} = \{F_z \subseteq [z, z+1): z \in Z\}$ of nonempty (or infinite) sets, but we can prove independence results if we require that at least one of the F_z be uncountable. In fact if we assume the continuum hypothesis, we can look at a much larger class of graphs. We define a set $S \subseteq R$ to be *strongly discrete* iff for any two distinct points $r, s \in S$, we have $|r - s| > 1$, and we let $\mathfrak{U}^D$ be the family of all strongly discrete subsets of R. Then we have:

THEOREM 4.1. *If the continuum hypothesis holds, there exists a* $\mathfrak{U}^D$*-graph*

$\langle R, \mathscr{G} \rangle$ such that for any uncountable set $U \subseteq R$, the set $R - \cup\{\boldsymbol{u}: u \in U\}$ is bounded.

PROOF. Let $\{r_\alpha: \alpha \in \aleph_1\}$ be an enumeration of R and let $\{C_\alpha: \alpha \in \aleph_1\}$ be an enumeration of the unbounded countable subsets of R. Then for each $\alpha \in \aleph_1$ we may, because the C_α are unbounded, choose a strongly discrete subset of R which does not contain r_α and which has nonempty intersection with each C_β such that $\beta < \alpha$. Call this set $\boldsymbol{r}_\alpha$, and the resulting graph $\mathscr{G}$.

Now let U be any subset of R such that the set $C = R - \cup\{\boldsymbol{u}: u \in U\}$ is unbounded. Then for some $\alpha \in \aleph_1$ we must have $C_\alpha \subseteq C$, so U must be a subset of $\{r_\beta: \beta \leqq \alpha\}$ and thus countable. ■

As with 3.7, we may substitute Martin's axiom for the continuum hypothesis in both the statement and the proof of this theorem. As before, we use the axiom to obtain the set $\boldsymbol{r}_\alpha$ which intersects each C_β such that $\beta < \alpha$ (even when α is uncountable). Again the set $\boldsymbol{r}_\alpha$ will be nowhere dense (we may use essentially the same construction as in 3.9), but will not have the additional properties which can be obtained given the continuum hypothesis. Thus we have:

COROLLARY 4.2. *If Martin's axiom holds, there exists a $\mathfrak{U}^N$-graph $\langle R, \mathscr{G} \rangle$ such that for any set $U \subseteq R$ of cardinality $2^{\aleph_0}$, the set $R - \cup\{\boldsymbol{u}: u \in U\}$ is bounded.* ■

Applying this to $\mathscr{Z}$ we have:

COROLLARY 4.3. *If Martin's axiom (the continuum hypothesis) holds, then there exists a $\mathfrak{U}^N-(\mathfrak{U}^D-)$ graph such that no infinite free family*

$$\mathscr{F} = \{F_z \subseteq [z, z+1): z \in Z\}$$

contains a member of cardinality $2^{\aleph_0}$. ■

On the other hand, not only is it consistent that there always exist such free families $\mathscr{F}$ *each* of whose members have cardinality $2^{\aleph_0}$, but it is consistent to assume in addition that $\cup\mathscr{F}$ be free. In fact we have the even stronger result:

THEOREM 4.4. *It is consistent with ZFC that every $\mathfrak{U}^N$-graph admit a free set F such that for every non-empty open interval (a,b), the set $F \cap (a,b)$ is of the second category and has cardinality $2^{\aleph_0}$.*

PROOF. It is known that the set G mentioned in the proof of 3.3 can be constructed to have the additional property that $|G \cap (a,b)| = 2^{\aleph_0}$ for every nonempty interval (a,b). The theorem then follows by applying 2.5 instead of 2.1 and letting $\mathscr{P} = \{G \cap (p,q): p, q \text{ rational}\}$. ■

If $\mathscr{E}$ is finite the situation is quite different. As our proofs will follow in spaces other than R with little or no extra difficulty, we shall consider the general case and begin with a slight generalization of the notion of category.

If $\mathscr{T} = \langle T, \mathscr{O} \rangle$ is any topological space, then the *weight* of $\mathscr{T}$ (which we shall denote by $wt(\mathscr{T})$) is defined to be the least cardinal λ such that $\mathscr{O}$ admits a base of cardinality λ. Using this, we define a set $S \subseteq T$ to be of the *first *category* iff it is a union of at most $wt(\mathscr{T})$ nowhere-dense subsets of T and to be of the second *category otherwise. We shall refer to sets of the first or second *category as *f-sets* or *s-sets* respectively. Finally, we define a set $S \subseteq T$ to be *categorically dense* over an open set O iff for every open set $P \subseteq O$, the set $S \cap P$ is an s-set. We note that if a set S is categorically dense over an open set O and P is any open subset of O, then S is categorically dense over P.

The reason we choose this definition of second *category is so that we might have:

LEMMA 4.5. *If $\mathscr{T} = \langle T, \mathscr{O} \rangle$ is any topological space and $S \subseteq T$ is any s-set, then there exists a set $O \in \mathscr{O}$ over which S is categorically dense.*

PROOF. Let $\mathscr{B}$ be a base for $\mathscr{O}$ of cardinality $\kappa = wt(\mathscr{T})$, let $\mathscr{F} = \{B \in \mathscr{B} : B \cap S$ is an f-set$\}$, and let $A = T - \cup\mathscr{F}$. Since $S \cap (\cup\mathscr{F})$ is a union of at most κ f-sets, it is itself an f-set, so $A \cap S$, and therefore A, must be an s-set. Thus, in particular, A cannot be nowhere dense. But since A is closed, it must have non-empty interior I, and it is easily seen that S is categorically dense over I. ∎

We shall also need:

LEMMA 4.6. *If A_0 and A_1 are any two disjoint s-sets in a Hausdorff space, then there exist disjoint open sets O_0 and O_1 such that each A_i (and therefore $A_i \cap O_i$) is categorically dense over O_i.*

PROOF. By 4.5 we may choose open sets O'_0 and O'_1 such that each $A_i \cap O'_i$ is categorically dense over O'_i. Choose any points p_0 and p_1 such that $p_i \in A_i \cap O_i$ and choose any pair O_0, O_1 of disjoint open sets such that $p_i \in O_i \subseteq O'_i$. ∎

And similarly:

LEMMA 4.7. *If A is any s-set in a Hausdorff space such that no member of A is an isolated point, then A may be split into two disjoint s-sets.*

PROOF. By 4.5 there exists an open set O over which A is categorically dense. Because A has no isolated points, we may choose two points $p_0, p_1 \in A \cap O$ and, because we are in a Hausdorff space, two disjoint open sets $O_0, O_1 \subseteq O$, such that $p_i \in O_i \subseteq O$. Clearly $A \cap O_0$ and $A - (A \cap O_0)$ are the two desired sets. ∎

We next consider finite families $\mathscr{E}$ of "reasonable" disjoint sets. From the following theorems we see that the proper definition of reasonable should be "s-set". We begin with a theorem which, although not stated in quite this form, is essentially due to Marcus [9].

THEOREM 4.8. *Let $\mathscr{T} = \langle T, \mathscr{O} \rangle$ be any Hausdorff space, let $\mathfrak{U}$ be the family of nowhere-dense subsets of T, and let $\mathscr{E} = \{E_i : i < n\}$ be any finite family of disjoint s-sets. Then every $\mathfrak{U}$-graph admits a free family $\{F_i \subseteq E_i : i < n\}$ of s-sets.*

PROOF. It is sufficient to prove the theorem for $n = 2$; the remainder will follow by induction. So suppose $\mathscr{E} = \{E_0, E_1\}$. By 4.6 we may assume that there exist disjoint open sets O_0 and O_1 such that each E_i is categorically dense over O_i. Let $\mathscr{P}$ be a base for $\mathscr{O}$ of cardinality $wt(\mathscr{T})$. Because $\langle T, \mathscr{G} \rangle$ is an $\mathfrak{U}$-graph, we may assign to each $a \in E_0$ a non-empty set $P_a \in \mathscr{P}$ such that $P_a \subseteq O_1$ and $a \cap P_a = \emptyset$. For each $P \in \mathscr{P}$ let $B_P = \{a \in E_0 : P_a = P\}$, and let $\mathscr{B} = \{B_P : P \in \mathscr{P}\}$. Since $|\mathscr{P}| = wt(\mathscr{T})$, we have $|\mathscr{B}| \leq wt(\mathscr{T})$. But $E_0 = \cup \mathscr{B}$ and E_0 is an s-set, so there must be at least one set $Q \in \mathscr{P}$ such that B_Q is an s-set.

Now, using 4.5, we choose a non-empty open set $O_0' \subseteq O_0$ over which B_Q is categorically dense, and we set $E_0' = B_Q \cap O_0'$, $E_1' = E_1 \cap Q$, and $O_1' = Q$. We note that as before, each E_i' is categorically dense over O_i', and $O_0' \cap O_1' = \emptyset$. However, we also have $(E_0' \times E_1') \cap \mathscr{G} = \emptyset$. Thus if we repeat our construction choosing $a \in E_1'$ and $P_a' \subseteq O_0$, the resulting s-sets E_0'' and E_1'' will be $\mathscr{G}$-disjoint. ∎

We return to our original question, that of the existence of infinite free families. As we have seen, we cannot hope to choose in advance the locations of the members of such families, but this does not mean such families do not exist. Not only do they exist, but we will even be able to specify the locations of their unions subject only to the conditions that these locations be s-sets and that they contain either infinitely many or no isolated points. This latter is necessary because, otherwise, the desired location might turn out to be the union of an f-set with a finite set of isolated points and, therefore, not contain any infinite family of disjoint s-sets.

THEOREM 4.9. *Let $\mathscr{T}$, $\mathfrak{U}$, and $\mathscr{G}$ be as in 4.8, and let $A \subseteq T$ be any s-set which contains either infinitely many isolated points or no isolated points. Then there exists a family $\mathscr{F} = \{F_i : i \in \omega\}$ of mutually $\mathscr{G}$-disjoint s-subsets of A.*

PROOF. If A contains an infinite set $\{a_i : i \in \omega\}$ of isolated points, let $\mathscr{F} = \{\{a_i\} : i \in \omega\}$. Otherwise we construct F_i inductively as follows. Assume we have already constructed a family $F_0, F_1, \cdots, F_{n-1}, A_n$ of mutually $\mathscr{G}$-disjoint infinite s-subsets of A. Then by 4.7, we may split A_n into two infinite s-sets B_0^n and B_1^n and then apply 4.8 to obtain two infinite $\mathscr{G}$-disjoint s-sets $F_n \subseteq B_0^n$ and $A_{n+1} \subseteq B_1^n$. The family $\mathscr{F} = \{F_i : i \in \omega\}$ now satisfies the theorem. ∎

If we look only at R, this result may be sharpened considerably. For any two sets A, $B \subseteq R$, define $A < B$ to hold iff for every $a \in A$ and $b \in B$, we have $a < b$. Then we have:

THEOREM 4.10. *Let $\langle R, \mathscr{G} \rangle$ be any $\mathfrak{U}^N$-graph over R and let $A \subseteq R$ be any s-set. Then there exists a family $\mathscr{F} = \{F_n : n \in \omega\}$ of mutually $\mathscr{G}$-disjoint s-sets such that $\cup \mathscr{F} \subseteq A$ and $\langle \mathscr{F}, < \rangle$ is a dense total-order structure with no first or last element.*

PROOF. We first note that if S is any s-set in R, then by applying twice the method used in proving 4.7 and then using 4.8, we can obtain three mutually $\mathscr{G}$-disjoint s-sets $S_0, S_1, S_2 \subseteq S$ such that $S_0 < S_1 < S_2$.

We first apply this on A to obtain three such mutually disjoint s-sets $A_0 < A_1 < A_2$. We then repeat this process with respect to A_0 and A_2 obtaining s-sets $A_3 < A_4 < A_5$ and $A_6 < A_7 < A_8$, but we leave A_1 fixed. This process is continued for all n always keeping sets of index $3i + 1$ fixed, and always partitioning each of the others into three subsets having the appropriate properties. We may now let $\mathscr{F} = \{A_{3i+1} : i \in \omega\}$. ∎

We now have the machinery available to prove the theorem mentioned just before 3.7, i.e. to prove that every $\mathfrak{U}^N$-graph over R admits free sets of every countable order type. We note that it is sufficient to prove that every such graph admits a free set of order type η, the order type of the rationals, because it is known that every countable order type can be embedded into η. But this we have already done; simply choose one point from each F_n belonging to the family $\mathscr{F}$ constructed in 4.10. Thus we have shown:

COROLLARY 4.11. *If $\langle R, \mathscr{G} \rangle$ is any $\mathfrak{U}^N$-graph, S is any s-set in R, and γ is any countable order type, then there exists a free-set $G \subseteq S$ which has order type γ under $<$.* ∎

We wish to thank Professor Erdös for suggesting this last question to us.

5. Open problems

We collect here some problems which we have mentioned earlier.

1. Does the existence of an $\mathfrak{U}^{NC}$-or $\mathfrak{U}^{N}$-graph which admits no uncountable free sets imply the continuum hypothesis?

2. Does Martin's axiom imply that every $\mathfrak{U}^{N}$-graph admits free sets of every cardinality less than $2^{\aleph_0}$?

3. Does every $\mathfrak{U}^{N}$-graph admit a dense or even non-nowhere dense, free set?

4. Does every $\mathfrak{U}^{N}$-graph admit a free family $\{F_z : z \in Z\}$ such that each F_z is a non-empty (infinite) subset of $[z, z+1]$?

5. Does every $\mathfrak{U}^{N}$-graph $\langle R, \mathscr{G} \rangle$ admit two (infinitely many) $\mathscr{G}$-disjoint sets each of cardinality $2^{\aleph_0}$?

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